

Quantum lower bound for sorting¹

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Abstract

We prove that $\Omega(n \log n)$ comparisons are necessary for any quantum algorithm that sorts n numbers with high success probability and uses only comparisons. If no error is allowed, at least $0.110n \log_2 n - 0.067n + O(1)$ comparisons must be made. The previous known lower bound is $\Omega(n)$.

Key words: sorting, quantum computation, decision tree complexity, lower bound.

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1 Introduction

The speedups of quantum algorithms over classical algorithms have been the main reason for current interests in quantum computation. Although dramatic speedups may be possible, for example, Shor's [16] algorithms for factoring and for finding discrete logarithms, the hunt for *provable* speedups has been successful only in very restricted models, such as the decision tree model.

In the decision tree model, the inputs x_1, x_2, \dots, x_n are known only to an oracle, and the only way that the algorithm can access the inputs is by asking the oracle questions of the type " $x_i = ?$ ". The complexity measurement is the number of such queries. Many problems that allow *provable* quantum speedups can be formulated in this model. For example, Grover's [14] algorithm for deciding i given the input e_i , i.e., the n -bits binary string that has a 1 in the i th position and 0 elsewhere, makes $O(\sqrt{n})$ queries, while any classical algorithm needs $\Omega(n)$ queries. Grover's algorithm has been the core quantum ingredient in the fast quantum decision tree algorithms for several other problems. Some examples are, the algorithm of Boyer et al. [8] for computing the *OR* of n Boolean inputs in $O(\sqrt{n})$ queries, the algorithm of Durr and Høyer [11] for finding the minimum of n numbers in $O(\sqrt{n})$ queries, and, more recently, the algorithm of Buhrman et al. [9] for solving the Element Distinctness problem in $O(n^{3/4} \log n)$ queries.

The optimality of Grover's algorithm is implied by the earlier result of Bennett et al. [6]. Beals et al. [4] and Ambainis [3] prove some powerful general lower bounds. Notably not all problems allow quantum speedups. The PARITY function is an example.

How fast can quantum computers sort? This is probably among the most natural questions to ask about the power of quantum computers. It is very natural to study the comparison-based sorting in the decision tree model. This is because the minimum number of comparisons needed to sort n numbers is just the same as the decision tree complexity of the following problem: given the comparison matrix of n numbers, output the order of these numbers. Without loss of generality, we assume that the comparison operator is " \leq ".

A straightforward information theoretical argument gives the $n \log_2 n$ classical lower bound, which is matched by simple sorting algorithms such as Insertion Sort with an $O(n)$ additive term. Using the quantum algorithm for searching an element in a sorted list by Farhi et al. [12] as a subroutine in Insertion Sort, a quantum computer needs only compare $0.526n \log_2 n + O(n)$ times. However, the hope of improving this upper bound asymptotically by improving the upper bound for the ordered search is made impossible

because of the $\Omega(\log n)$ lower bound for the latter problem due to Ambainis [2].

Ambainis' lower bound proof for the ordered search does not seem to apply to the sorting problem. However, a straightforward application of the same author's general lower bound technique [3] gives an $\Omega(n)$ lower bound. In this paper, we improve the lower bound to $\Omega(n \log n)$. This means that the best comparison-based quantum sorting algorithm can be at most a constant time faster than the best classical algorithm. Our approach is similar to that taken by Høyer and Nerbeek [15] in improving the lower bound for the ordered search by a constant factor. That is, we use the *weighted* version of the adversary idea of Ambainis [3] with a carefully chosen probabilistic distribution of adversary input pairs, and the lower bound finally relies on a property of the Hilbert matrix.

In the next section we give a formal definition of the quantum decision tree model. After introducing some notations, we give a proof overview. The complete proof of the main lemma is presented in Section 3, which is followed by the section for open problems.

2 Notations and overview

2.1 The model

Following the paper of Beals et al. [4], we give a formal definition of the standard quantum decision tree model for computing a function $f : \{0, 1\}^m \rightarrow \{0, 1\}^k$.

The quantum algorithm works in the Hilbert space

$$\text{span}\{|s, b, c\rangle : s \in \mathbb{Z}_m, b \in \{0, 1\}, c \in \{0, 1\}^w, \text{ for some integer } w \geq 0\}.$$

An oracle holds the input $x = x_0 x_1 \cdots x_{m-1} \in \{0, 1\}^m$, which is not known to the algorithm directly. Recall that in the classical decision tree model, the input bit x_i is returned to the algorithm if the index i is presented to the oracle. In the quantum setting, due to the requirement of being unitary for all quantum operations, x_i is XOR-ed with another bit provided by the algorithm. Mathematically, the algorithm “reads” the input through the *oracle gate*, which is a unitary transformation that depends on x and works on each base vector $|s, b, c\rangle$ as follows:

$$O_x |s, b, c\rangle = |s, b \oplus x_s, c\rangle.$$

It is easy to verify that with $\Psi^\pm = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$,

$$O_x |s, \Psi^-, c\rangle = (-1)^{x_s} |s, \Psi^-, c\rangle,$$

and,

$$O_x|s, \Psi^+, c\rangle = |s, \Psi^+, c\rangle.$$

A quantum decision tree algorithm that makes T queries is the application of a sequence of unitary transformations on the initial state $|\vec{0}\rangle$:

$$U_T O_x U_{T-1} O_x \cdots U_1 O_x U_0 |\vec{0}\rangle.$$

For all t , $0 \leq t \leq T$, we write

$$\phi_x^{(t)} := U_t O_x \cdots U_0 |\vec{0}\rangle.$$

We say that the quantum algorithm computes f with the error probability bounded by ϵ , for some constant ϵ such that $0 \leq \epsilon < 1/2$, if there is a measurement M , such that for any input x , when M is applied to $\phi_x^{(T)}$, the probability of observing $f(x)$ is no less than $1 - \epsilon$.

2.2 More notations

For our lower bound purpose, it suffices to assume that x_0, x_1, \dots, x_{n-1} , the n numbers to be sorted, correspond to some permutation σ of $\{0, 1, \dots, n-1\}$, i.e., $x_i = \sigma(i)$, for all $0 \leq i \leq n-1$. The input to the quantum decision tree algorithm is the comparison matrix $M_\sigma \in \mathbf{M}_n$, i.e., for all i and j , $0 \leq i, j \leq n-1$,

$$(M_\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(i) \leq \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

For the simplicity of notations, when the subscript M_σ is needed, we use σ instead. The Hilbert space of the algorithm is now

$$H := \text{span}\{|i, j, b, c\rangle : 0 \leq i, j \leq n-1, b \in \{0, 1\}, c \in \{0, 1\}^w, \text{ for some } w \geq 0\}.$$

The oracle gate that corresponds to the input σ works as follows:

$$O_\sigma|i, j, b, c\rangle = |i, j, b \oplus (M_\sigma)_{i,j}, c\rangle.$$

For all i and j , $0 \leq i, j \leq n-1$, let $P_{i,j}$ be projection from H to the subspace

$$\text{span}\{|i, j, \Psi^-, c\rangle, |j, i, \Psi^-, c\rangle : c \in \{0, 1\}^w\}.$$

For any permutation σ , and any $k, d \in \mathbf{Z}$ with $0 \leq k \leq n-2$ and $1 \leq d \leq n-k-1$, we define a new permutation

$$\sigma^{(k,d)} := (k+d, k+d-1, \dots, k) \circ \sigma.$$

Note that if $\tau = \sigma^{(k,d)}$, we have

$$\sigma^{-1}(i) = \begin{cases} \tau^{-1}(i+d) & i = k, \\ \tau^{-1}(i-1) & k+1 \leq i \leq k+d, \\ \tau^{-1}(i) & \text{otherwise.} \end{cases}$$

Also note that M_σ and M_τ differ on only the following pairs of indices

$$\{\sigma^{-1}(k), \sigma^{-1}(k+i)\} = \{\tau^{-1}(k+d), \tau^{-1}(k+i-1)\},$$

for all i such that $1 \leq i \leq d$. The weight function $w(\sigma, \tau)$ is defined for every pair of permutations:

$$w(\sigma, \tau) := \begin{cases} 1/d & \text{if } \tau = \sigma^{(k,d)}, \text{ for some } k \text{ and } d, \\ & \text{such that } 0 \leq k \leq n-2 \text{ and } 1 \leq d \leq n-k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $H_n := \sum_{i=1}^n 1/i$.

2.3 Proof overview

Now we describe the main idea of the adversary technique. For any pair of the inputs (x, y) such that $f(x) \neq f(y)$, any algorithm that computes f with high success probability must separate the two final vectors $\phi_x^{(T)}$ and $\phi_y^{(T)}$ far apart. However, it is hard for the algorithm to distinguish x and y if they are very similar. Therefore, if there is a probabilistic distribution of close pairs so that on average the algorithm can only separate the corresponding vectors by a little amount on each step, then we can argue that the algorithm needs many steps. The quantity $|\langle \phi_x^{(t)} | \phi_y^{(t)} \rangle|$ is used to measure how close the two vectors are.

The follow quantity is an indication of the algorithm's progress:

$$s_t := \sum_{\sigma, \tau} w(\sigma, \tau) \left| \langle \phi_\sigma^{(t)} | \phi_\tau^{(t)} \rangle \right|.$$

The idea behind the choice of $w(\sigma, \tau)$ is that, for any permutation σ , these permutations τ obtained from σ by rotating some d consecutive elements are close to σ . Moreover, the smaller d is, the harder for the algorithm to distinguished them. Therefore $w(\sigma, \tau)$ is proportional to $1/d$.

Since the algorithm starts with the same initial state regardless of the input, by usual calculation we have,

Lemma 2.1 $s_0 = n!(nH_{n-1} - n + 1)$.

It is a well known fact that if an algorithm computes a function f with the error probability bounded by ϵ , then for any pair of inputs x and y such that $f(x) \neq f(y)$,

$$|\langle \phi_x^{(T)} | \phi_y^{(T)} \rangle| \leq 2\sqrt{\epsilon(1-\epsilon)}.$$

See, for example, the paper of Ambainis [3] for a proof. Henceforth,

Lemma 2.2 $s_T \leq 2\sqrt{\epsilon(1-\epsilon)}s_0$.

Our main lemma says that any algorithm can make only a little bit of progress in decreasing s_t on each step:

Lemma 2.3 (Main Lemma) $|s_{t+1} - s_t| \leq 2\pi n!$, for all $0 \leq t \leq T - 1$.

Since

$$(1 - 2\sqrt{\epsilon(1-\epsilon)})s_0 \leq |s_T - s_0| \leq \sum_{t=0}^{T-1} |s_{t+1} - s_t|,$$

by the previous lemmas, we obtain our main theorem:

Theorem 2.4 (Main Theorem) *Any quantum sorting algorithm with the error probability bounded by ϵ must compare no less than*

$$\frac{1 - \sqrt{2\epsilon(1-\epsilon)}}{2\pi}(nH_{n-1} - n + 1) = \Omega(n \log n)$$

times.

Setting $\epsilon = 0$, we obtain

Corollary 2.5 *Any error-less quantum sorting algorithm must compare at least*

$$\frac{1}{2\pi}n \ln n - \frac{1 - C_E}{2\pi}n + O(1) \approx 0.110n \log_2 n - 0.067n + O(1)$$

times, where $C_E = 0.57721566 \dots$ is the Euler-Mascheroni Constant.

Let $A = [\alpha_{k,l}]_{1 \leq k, l < \infty}$ be the Hilbert matrix with $\alpha_{k,l} = 1/(k + l - 1)$, and $\|\cdot\|_2$ be the spectral norm, i.e., for any complex matrix $M \in \mathbf{M}_m$,

$$\|M\|_2 := \max_{x \in \mathbb{C}^m, \|x\|_2=1} \|Mx\|_2.$$

Our proof for the Main Lemma relies on the following property of the Hilbert matrix:

Lemma 2.6 $\|A\|_2 = \pi$.

Choi [10] has an elegant proof for this lemma.

3 Proof of the Main Lemma

Proof. [Lemma 2.3] For each individual pair (σ, τ) , where $\tau = \sigma^{(k,d)}$, by the definitions of $\phi_\sigma^{(t+1)}$ and $\phi_\tau^{(t+1)}$, we have

$$\begin{aligned} & \left| \langle \phi_\sigma^{(t+1)} | \phi_\tau^{(t+1)} \rangle - \langle \phi_\sigma^{(t)} | \phi_\tau^{(t)} \rangle \right| \\ = & \left| \left\langle U_{t+1}(O_\sigma \phi_\sigma^{(t)} - O_\tau \phi_\sigma^{(t)}) + U_{t+1} O_\tau \phi_\sigma^{(t)} \middle| U_{t+1} O_\tau \phi_\tau^{(t)} \right\rangle - \langle \phi_\sigma^{(t)} | \phi_\tau^{(t)} \rangle \right|. \end{aligned}$$

By the properties of the inner product and that U_{t+1} is unitary, the above expressions are simplified to

$$\left| \langle (O_\tau O_\sigma - I) \phi_\sigma^{(t)} | \phi_\tau^{(t)} \rangle \right|. \quad (1)$$

The effects of O_σ and O_τ cancel out on most base vectors, except for these $|i, j, \Psi^-, c\rangle$ such that $(M_\sigma)_{i,j} \neq (M_\tau)_{i,j}$. Therefore Eq. 1 is bounded from the above by

$$2 \sum_{i=1}^d \left| \langle P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_\sigma^{(t)} | \phi_\tau^{(t)} \rangle \right|.$$

By the Cauchy-Schwarz inequality, this expression can be further upper-bounded by

$$2 \sum_{i=1}^d \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_\sigma^{(t)} \right\| \cdot \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_\tau^{(t)} \right\|. \quad (2)$$

Now we are ready to bound $\Delta_t := |s_{t+1} - s_t|$. By definitions and pulling summations out of the absolute value, we obtain

$$\Delta_t \leq \sum_{\sigma} \sum_{k=0}^{n-2} \sum_{d=1}^{n-k-1} \frac{1}{d} \left| \langle \phi_\sigma^{(t+1)} | \phi_{\sigma^{(k,d)}}^{(t+1)} \rangle - \langle \phi_\sigma^{(t)} | \phi_{\sigma^{(k,d)}}^{(t)} \rangle \right|. \quad (3)$$

Plug in the upper bound of Eq. 2, this is then upper-bounded by

$$2 \sum_{\sigma} \sum_{k=0}^{n-2} \sum_{d=1}^{n-k-1} \sum_{i=1}^d \frac{1}{d} \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_\sigma^{(t)} \right\| \cdot \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_{\sigma^{(k,d)}}^{(t)} \right\|.$$

By reordering the terms of the summation and applying the Cauchy-Schwarz inequality, this is further upper-bounded by

$$2 \sum_{d=1}^{n-1} \frac{1}{d} \sum_{i=1}^d \sqrt{\sum_{\sigma} \sum_{k=0}^{n-d-1} \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_\sigma^{(t)} \right\|^2} \sqrt{\sum_{\sigma} \sum_{k=0}^{n-d-1} \left\| P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_{\sigma^{(k,d)}}^{(t)} \right\|^2}. \quad (4)$$

Let $a = [a_i]_{1 \leq i \leq n-1} \in \mathbb{C}^{n-1}$ be a column vector with

$$a_i := \sqrt{\sum_{\sigma} \sum_{l=0}^{n-i-1} \|P_{\sigma^{-1}(l), \sigma^{-1}(l+i)} \phi_{\sigma}^{(t)}\|^2}.$$

Clearly,

$$\sqrt{\sum_{\sigma} \sum_{k=0}^{n-d-1} \|P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_{\sigma}^{(t)}\|^2} \leq a_i,$$

and,

$$\begin{aligned} & \sqrt{\sum_{\sigma} \sum_{k=0}^{n-d-1} \|P_{\sigma^{-1}(k), \sigma^{-1}(k+i)} \phi_{\sigma^{(k,d)}}^{(t)}\|^2} \\ &= \sqrt{\sum_{\tau} \sum_{k=0}^{n-d-1} \|P_{\tau^{-1}(k+d), \tau^{-1}(k+i-1)} \phi_{\tau}^{(t)}\|^2} \\ &\leq a_{d-i+1}. \end{aligned}$$

Let $K_{n-1} = [\kappa_{k,l}]_{1 \leq k, l \leq n-1} \in \mathbf{M}_{n-1}$ be a Hankel matrix with

$$\kappa_{k,l} = \begin{cases} 1/(k+l-1) & k+l \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Now Eq. 4 can be upper-bounded by

$$2 \sum_{d=1}^{n-1} \sum_{i=1}^d \frac{1}{d} a_i a_{d-i+1} = 2a^T K_{n-1} a. \quad (5)$$

Since every $\phi_{\sigma}^{(t)}$ is a unit vector, we have

$$\begin{aligned} \|a\|_2^2 &= \sum_{i=1}^{n-1} \sum_{\sigma} \sum_{l=0}^{n-i-1} \|P_{\sigma^{-1}(l), \sigma^{-1}(l+i)} \phi_{\sigma}^{(t)}\|^2 \\ &= \sum_{\sigma} \sum_{l=0}^{n-2} \sum_{i=1}^{n-l-1} \|P_{\sigma^{-1}(l), \sigma^{-1}(l+i)} \phi_{\sigma}^{(t)}\|^2 \\ &\leq n!. \end{aligned}$$

Clearly $\|K_{n-1}\|_2 \leq \|A\|_2 = \pi$, by Lemma 2.6. Applying the Cauchy-Schwarz inequality and by the definition of the spectral norm, we obtain the desired upper bound:

$$|s_{t+1} - s_t| \leq 2\|a\|_2^2 \|K_{n-1}\|_2 \leq 2\pi n!.$$

■

4 Open problems

The result of Grigoriev et al. [13] implies that if only comparisons are allowed, the randomized decision tree complexity of Element Distinctness has the same $\Omega(n \log n)$ lower bound as the sorting problem. Interestingly, their quantum complexities differ dramatically: the quantum algorithm for Element Distinctness due to Buhrman et al. [9] compares only $O(n^{3/4} \log n)$ times. It would be interesting to improve the current trivial lower bound of $\Omega(\sqrt{n})$ for Element Distinctness.

The decision tree complexity of nontrivial monotone graph properties has been a classical subject. Recently, Yao [17] proves an $\Omega(n^{2/3})$ lower bound for the quantum complexity of every nontrivial monotone graph property. It is reasonable to conjecture that the correct general quantum lower bound should be $\Omega(n)$. The best known lower bounds for specific properties, such as the $\Omega(n)$ lower bound for Connectivity implied by Ambainis' [3] general technique, do not seem to be tight. It would be interesting to improve both the general and the specific lower bounds. Probably this will require new techniques other than that of Ambainis [3].

Space-time tradeoffs for sorting and related problems have been studied for the classical case. A *Time · Space* lower bound of $\Omega(n^2)$ is proved for the comparison-based sorting by Borodin et al. [7], and for the R-way branching program by Beame [5]. Formulations and results on the quantum time-space tradeoffs for sorting and other problems such as Element Distinctness would be interesting.

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